Semidefinite programming bounds for codes and anticodes in Cayley graphs

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Theory
Codes and anticodes in Cayley graphs

\[ \text{Cayley}(G, \Sigma) \quad x \sim y \iff xy^{-1} \in \Sigma \]

\[ \uparrow \uparrow \]

\text{group} \quad \Sigma \subseteq G, \Sigma = \Sigma^{-1}

undirected graph on \( G \)

may contain loops

\( I \subseteq G \) independent: \( \forall x, y \in I, x \neq y, x \not\sim y \)

find indep. sets in \( \text{Cayley}(G, \Sigma) \)

which are as “large” as possible

max. packing density: \( \bar{\alpha}(\text{Cayley}(G, \Sigma)) \)

\( \bar{\alpha} = 2/5 \)

\( \text{Cayley}(\mathbb{Z}/5\mathbb{Z}, \{1, 4\}) \)
Examples

a) $k$-intersecting permutations
   \[ G = S_n, \quad \Sigma = \{\sigma : \sigma \text{ has } < k \text{ fixed points}\} \]

b) $k$-intersecting transformations
   \[ G = \text{GL}(n, \mathbb{F}_q), \quad \Sigma = \{A : \text{rank}(A - I) > n - k\} \]

c) distance-1-avoiding sets
   \[ G = \mathbb{R}^n, \quad \Sigma = S^{n-1} \]

d) sphere packings
   \[ G = \mathbb{R}^n, \quad \Sigma = B^\circ_n \]

e) packing of congruent convex bodies
   \[ G = \mathbb{R}^n \rtimes \text{SO}(n), \quad \Sigma = \{(x, A) : K^\circ \cap x + AK^\circ \neq \emptyset\} \]
Known results

a), b) optima realized by ”sunflowers”

$$I = \{ \sigma : \sigma(1) = 1, \ldots, \sigma(k) = k \}$$

proved (for $n$ large wrt. $k$) by Ellis, Friedgut, Pilpel (2011)

$$I = \{ A : Ae_1 = e_1, \ldots, Ae_k = e_k \}$$

conjectured by DeCorte, de Laat, V. (2013)
c)—e) wide open

c) closely related: chromatic number of the plane

d) only known for $n = 2, 3$

e) $\mathcal{K} = \text{regular tetradedron} \quad \overline{\alpha} \in [0.85, 1 - 10^{-26}]$

Chen, Engel, Glotzer (2010)
Gravel, Elser, Kallus (2011)

$\mathcal{K} = \text{regular pentagon} \quad \overline{\alpha} \in [0.92, ?]$

Kuperberg$^2$ (1992)
Bounds

a)–e) upper bound come from spectral techniques (convex optimization & harmonic analysis)

distinction between coding and anticoding problems

\[
\begin{array}{c}
\{ \text{anticoding} \\
\text{coding} \}
\end{array}
\]

problem: if

\[
\begin{array}{c}
e \notin \Sigma \\
e \in \Sigma
\end{array}
\]

packing of point measures vs. continuous measures
Examples

a) $k$-intersecting permutations
\[ G = S_n, \ \Sigma = \{ \sigma : \sigma \text{ has } < k \text{ fixed points} \} \]

b) $k$-intersecting transformations
\[ G = \text{GL}(n, \mathbb{F}_q), \ \Sigma = \{ A : \text{rank}(A - I) > n - k \} \]

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d) sphere packings
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e) packing of congruent convex bodies
\[ G = \mathbb{R}^n \rtimes \text{SO}(n), \ \Sigma = \{ (x, A) : \mathcal{K}^\circ \cap x + A\mathcal{K}^\circ \neq \emptyset \} \]
anticodes:

$$\bar{\alpha} \leq \sup \left\{ \frac{\int_G f(x) \, d\mu(x)}{f(e)} : f : G \to \mathbb{R} \text{ pos. type} \right\}$$

$$f(x) = 0 \text{ if } x \in \Sigma$$

\[f\] positive type:

$$\forall x_1, \ldots, x_N \in G : \left( f(x_i x_j^{-1}) \right)_{1 \leq i, j \leq N} \text{ is pos. semidefinite}$$

if \( G \) finite, then optimal solution is Lovász' \( \vartheta(G) \)

If \( I \subseteq G \) indep., then \( 1_I * \tilde{1}_I(x) = \int_G 1_I(y)1_I(y x^{-1}) \, d\mu(y) \)

is feasible
anticodes:

\[
\overline{\alpha} \leq \sup \left\{ \frac{\int_G f(x) \, d\mu(x)}{f(e)} : f : G \to \mathbb{R} \text{ pos. type} \right. \\
\left. f(x) = 0 \text{ if } x \in \Sigma \right\}
\]

codes:

\[
\overline{\alpha} \leq \inf \left\{ \frac{f(e)}{\int_G f(x) \, d\mu(x)} : f : G \to \mathbb{R} \text{ pos. type} \right. \\
\left. f(x) \leq 0 \text{ if } x \notin \Sigma \right\}
\]

if \( G = \mathbb{F}_q^n \), then optimal solution is Delsarte’s LP bound
Computing the bounds

* parametrize cone of positive type functions & use conic optimization

construction of positive type functions

\[ \pi : G \to U(H_\pi) \] unitary representation, \( h \in H_\pi \)

then \( f(x) = (\pi(x)h, h) \) is positive type

* Gelfand-Raikov 1942:
  * all positive type functions are of this form
  * extreme rays of cone of pos. type functions come from irreducible rep.
Segal-Mautner 1950:

If $G$ is nice and if $f$ is rapidly decreasing:

\[ f(x) = \int_{\hat{G}} \text{trace}(\pi(x) \hat{f}(\pi)) \, d\nu(\pi) \]

for positive, trace-class operators $\hat{f}(\pi) : H_\pi \to H_\pi$

\[ \hat{G} = \{ \text{irred. unitary rep. of } G \} / \sim \]

$\nu = \text{Plancherel measure on } \hat{G}$

\[ \hat{f}(\pi) = \int_{G} f(x) \pi(x^{-1}) \, d\mu(x) \]

Fourier transform
a)—d) \( \Sigma \) closed under conjugation
\[ \implies \text{can restrict to central pos. type functions} \]

\( f \) central: \( f(xy) = f(yx) \)

\[ f(x) = \int_{\hat{G}} \chi_\pi(x) \bar{f}(\pi) \, d\nu(\pi) \]

\( \chi_\pi \) irreducible character

\[ \bar{f}(\pi) \geq 0 \quad \forall \pi \in \hat{G} \]

\( \star \) SDP collapses to LP
\( \star \) can be analyzed by hand for a), c)
\( b) \) not yet
\( \star \) d) Cohn-Elkies (2003) LP bound
e) relevant irred. rep. of $\mathbb{R}^n \rtimes \text{SO}(n)$

$$\pi_a : G \to \text{U}(L^2(S^1)) \quad a > 0$$

$$[\pi_a(x, A)\varphi](\xi) = e^{2\pi i a x \cdot \xi} \varphi(A^{-1}\xi)$$

$$f(x, A) = 2\pi \int_0^\infty \text{trace}(\pi_a(x, A)\hat{f}(a)) a \, da$$

in polar coordinates

$$f(\rho, \theta, \alpha) = \int_0^\infty \sum_{r,s \in \mathbb{Z}} \hat{f}(a)_{r,s} i^{s-r} e^{-i(s\alpha + (r-s)\theta)} J_{s-r}(2\pi a \rho) a \, da$$

$$x = \rho(\cos \theta, \sin \theta), \quad A = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}$$
Explicit computations

the problem of finding an optimal function is an infinite-dimensional SDP

goal: reformulate and relax to a finite-dimensional SDP

solve this rigorously on a computer
When
\[ \hat{f}(a)_{r,s} = \sum_{k=0}^{d} f_{r,s;k} a^{2k} e^{-\pi a^2} \]
and setting the right \( \hat{f}(a)_{r,s} \) to zero forces
\[ f(\rho, \theta, \alpha) = \int_{0}^{\infty} \sum_{r,s \in \mathbb{Z}} \hat{f}(a)_{r,s} i^{s-r} e^{-i(s\alpha+(r-s)\theta)} J_{s-r}(2\pi a\rho) a \, da \]
to become a polynomial times exponential.

If
\[ e^{\pi a^2} \sum_{r,s=-N}^{N} \hat{f}(a) y_r y_s \in \mathbb{R}[a, y_{-N}, \ldots, y_N] \]
is a sum of squares, then \( f \) is pos. type.
geometric condition

\[ f(x, A) \leq 0 \text{ if } x \not\in K - AK \]
complete SDP (with only a few minor mistakes)
complete SDP (with only a few minor mistakes)
continued
Kuperberg\textsuperscript{2} (1992)
\[
\bar{\alpha} \in [0.92, \text{?}]
\]

0.98 Oliveira, V. (2013)

- custom made C++ library for generating and analyzing SDPs with SOS constraints
- geometric constraint modeled by a mixture of sampling and SOS
- 0.98 can probably be improved
Improving Cohn-Elkies bound

de Laat, Oliveira, V. (2012)

1. Adding valid inequalities
   (bounds on average contact numbers)

2. More flexible numerical method

<table>
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<th>Rogers</th>
<th>Cohn-Elkies</th>
<th>new bound</th>
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</table>

density given as point density (= # centers per unit volume)
Rigorous computations

right choice of polynomial basis is extremely important

— using monomial basis fails badly, even for very small degrees

— our choice: \( \mu_k L_{n/2}^{-1}(2\pi t) \)

\( \mu_k \): coefficient of \( L_{n/2}^{-1}(2\pi t) \) with largest absolute value

— csdp: \( d \leq 31 \)

— SDPA-gmp with 256 bits of precision: \( d \leq 51 \)
In order to get mathematical rigorous results:

— perform post processing of the floating point solution

— perturb to a rational solution

— analyze quality-loss of this perturbation

(by estimates of eigenvalues and condition numbers)
Tetrahedra?

★ needs more automatization
   (also the harmonic analysis part)

★ needs more theory for numerical optimization with SOS constraints
   (condition numbers, special numerical solvers)

★ still a challenge
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