Semidefinite programming bounds for codes and anticodes in Cayley graphs

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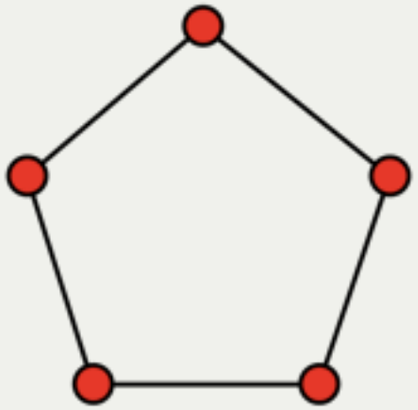
Theory

Codes and anticodes in Cayley graphs

$\text{Cayley}(G, \Sigma)$ $\qquad x \sim y \iff xy^{-1} \in \Sigma$

group $\quad \Sigma \subseteq G, \Sigma = \Sigma^{-1}$

undirected graph on G
may contain loops



$\overline{\alpha} = 2/5$

$\text{Cayley}(\mathbb{Z}/5\mathbb{Z}, \{1, 4\})$

$I \subseteq G$ independent: $\forall x, y \in I, x \neq y, x \not\sim y$

find indep. sets in $\text{Cayley}(G, \Sigma)$
which are as "large" as possible

max. packing density: $\overline{\alpha}(\text{Cayley}(G, \Sigma))$

Examples

a) k-intersecting permutations

$G = S_n,\ \Sigma = \{\sigma : \sigma \text{ has} < k \text{ fixed points}\}$

b) k-intersecting transformations

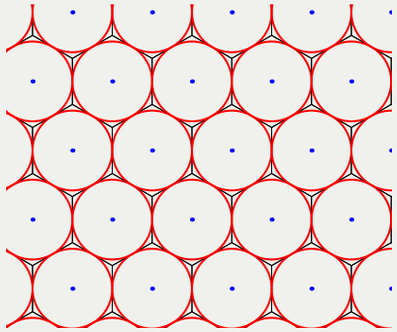
$G = \mathsf{GL}(n, \mathbb{F}_q),\ \Sigma = \{A : \text{rank}(A - I) > n - k\}$

c) distance-1-avoiding sets

$G = \mathbb{R}^n,\ \Sigma = S^{n-1}$

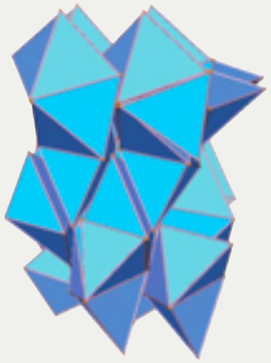
d) sphere packings

$G = \mathbb{R}^n,\ \Sigma = B_n^\circ$



e) packing of congruent convex bodies

$G = \mathbb{R}^n \rtimes \mathsf{SO}(n),\ \Sigma = \{(x, A) : \mathcal{K}^\circ \cap x + A\mathcal{K}^\circ \neq \emptyset\}$



difficulty

Known results

a), b) optima realized by "sunflowers"

$$I = \{\sigma : \sigma(1) = 1, \ldots, \sigma(k) = k\}$$

proved (for n large wrt. k) by Ellis, Friedgut, Pilpel (2011)

$$I = \{A : Ae_1 = e_1, \ldots, Ae_k = e_k\}$$

conjectured by DeCorte, de Laat, V. (2013)

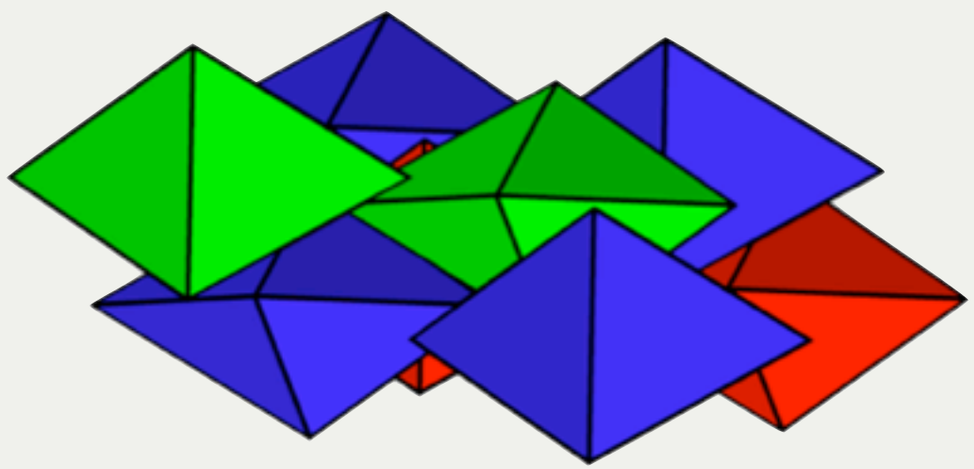
	$q = 2$						$q = 3$				$q = 4$		
k \ n	1	2	3	4	5	6	1	2	3	4	1	2	3
1	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
2		✓	✓		✓	✓		✓	✓	✓		✓	✓
3			✓	✓		✓			✓	✓			✓
4				✓	✓					✓			
5					✓	✓							
6						✓							

c)—e) wide open

c) closely related: chromatic number of the plane

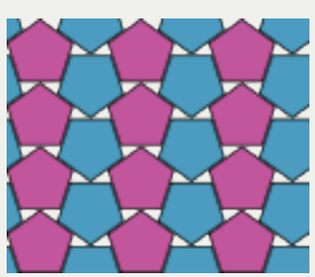
d) only known for $n = 2, 3$

e) $\mathcal{K}$ = regular tetradedron $\quad \overline{\alpha} \in [0.85, 1 - 10^{-26}]$



Chen, Engel, Glotzer (2010)
Gravel, Elser, Kallus (2011)

$\mathcal{K}$ = regular pentagon $\quad \overline{\alpha} \in [0.92, ?]$



Kuperberg[2] (1992)

Bounds

a)–e) upper bound come from spectral techniques
(convex optimization & harmonic analysis)

distinction between coding and anticoding problems

$$\left\{ \begin{array}{c} \text{anticoding} \\ \text{coding} \end{array} \right\} \text{ problem: if } \left\{ \begin{array}{c} e \notin \Sigma \\ e \in \Sigma \end{array} \right\}$$

packing of point measures vs. continuous measures

Examples

a.c.

a) k-intersecting permutations

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c.

d) sphere packings

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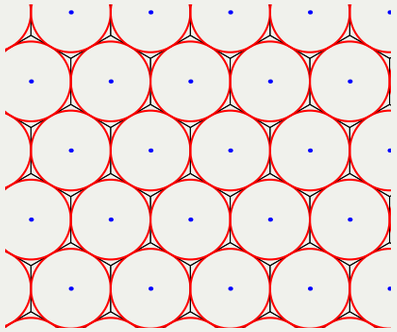
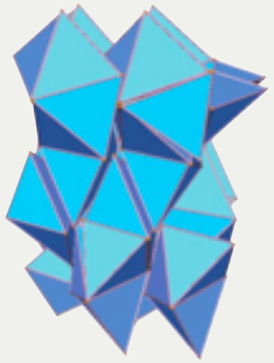




e) packing of congruent convex bodies

$G = \mathbb{R}^n \rtimes \mathsf{SO}(n),\ \Sigma = \{(x, A) : \mathcal{K}^\circ \cap x + A\mathcal{K}^\circ \neq \emptyset\}$

anticodes:

$$\overline{\alpha} \leq \sup \left\{ \frac{\int_G f(x)\, d\mu(x)}{f(e)} : \begin{array}{l} f : G \to \mathbb{R} \text{ pos. type} \\ f(x) = 0 \text{ if } x \in \Sigma \end{array} \right\}$$

f positive type:

$\forall x_1, \ldots, x_N \in G : \left(f(x_i x_j^{-1}) \right)_{1 \leq i,j \leq N}$ is pos. semidefinite

if G finite, then optimal solution is Lovász' $\vartheta(G)$

If $I \subseteq G$ indep., then $1_I * \tilde{1}_I(x) = \int_G 1_I(y) 1_I(yx^{-1}) d\mu(y)$

is feasible

$\tilde{f}(x) = \overline{f(x^{-1})}$

anticodes:

$$\overline{\alpha} \leq \sup \left\{ \frac{\int_G f(x)\, d\mu(x)}{f(e)} \; : \; \begin{array}{l} f : G \to \mathbb{R} \text{ pos. type} \\ f(x) = 0 \text{ if } x \in \Sigma \end{array} \right\}$$

codes:

$$\overline{\alpha} \leq \inf \left\{ \frac{f(e)}{\int_G f(x)\, d\mu(x)} \; : \; \begin{array}{l} f : G \to \mathbb{R} \text{ pos. type} \\ f(x) \leq 0 \text{ if } x \notin \Sigma \end{array} \right\}$$

if $G = \mathbb{F}_q^n$, then optimal solution is Delsarte's LP bound

Computing the bounds

- parametrize cone of positive type functions
  & use conic optimization

construction of positive type functions

$\pi : G \to U(H_\pi)$ unitary representation, $h \in H_\pi$
then $f(x) = (\pi(x)h, h)$ is positive type

- Gelfand-Raikov 1942:
  - all positive type functions are of this form
  - extreme rays of cone of pos. type functions
    come from irreducible rep.

Segal-Mautner 1950:

If G is nice and if f is rapidly decreasing:

f is pos. type $\iff$

optimization variable

$$f(x) = \int_{\widehat{G}} \operatorname{trace}(\pi(x)\hat{f}(\pi))\, d\nu(\pi)$$

for positive, trace-class operators $\widehat{f}(\pi) : H_\pi \to H_\pi$

$\widehat{G} = \{\text{irred. unitary rep. of } G\}/\sim$

$\nu =$ Plancherel measure on $\widehat{G}$

$\hat{f}(\pi) = \int_G f(x)\pi(x^{-1})\, d\mu(x)$ Fourier transform

a)—d) Σ closed under conjugation

$\Longrightarrow$ can restrict to central pos. type functions

f central: $f(xy) = f(yx)$

$$f(x) = \int_{\widehat{G}} \chi_\pi(x) \bar{f}(\pi)\, d\nu(\pi)$$

χ_π irreducible character

$$\bar{f}(\pi) \geq 0 \; \forall \pi \in \widehat{G}$$

- ⋆ SDP collapses to LP
- ⋆ can be analyzed by hand for a), c)

  b) not yet
- ⋆ d) Cohn-Elkies (2003) LP bound

e) relevant irred. rep. of $\mathbb{R}^n \rtimes \mathsf{SO}(n)$

$$\pi_a : G \to \mathsf{U}(L^2(S^1)) \qquad a > 0$$

$$[\pi_a(x, A)\varphi](\xi) = e^{2\pi i a x \cdot \xi} \varphi(A^{-1}\xi)$$

$$f(x, A) = 2\pi \int_0^\infty \mathrm{trace}(\pi_a(x, A)\hat{f}(a)) a \, da$$

in polar coordinates

$$f(\rho, \theta, \alpha) = \int_0^\infty \sum_{r,s \in \mathbb{Z}} \hat{f}(a)_{r,s} i^{s-r} e^{-i(s\alpha + (r-s)\theta)} J_{s-r}(2\pi a \rho) a \, da$$

$$x = \rho(\cos\theta, \sin\theta), \; A = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

Explicit computations

the problem of finding an optimal function is an infinite-dimensional SDP

goal: reformulate and relax to a finite-dimensional SDP

solve this <u>rigorously</u> on a computer

When

$$\hat{f}(a)_{r,s} = \sum_{k=0}^{d} f_{r,s;k} a^{2k} e^{-\pi a^2}$$

and setting the right $\hat{f}(a)_{r,s}$ to zero forces

$$f(\rho,\theta,\alpha) = \int_0^\infty \sum_{r,s\in\mathbb{Z}} \hat{f}(a)_{r,s} i^{s-r} e^{-i(s\alpha+(r-s)\theta)} J_{s-r}(2\pi a\rho) a\, da$$

to become a polynomial times exponential

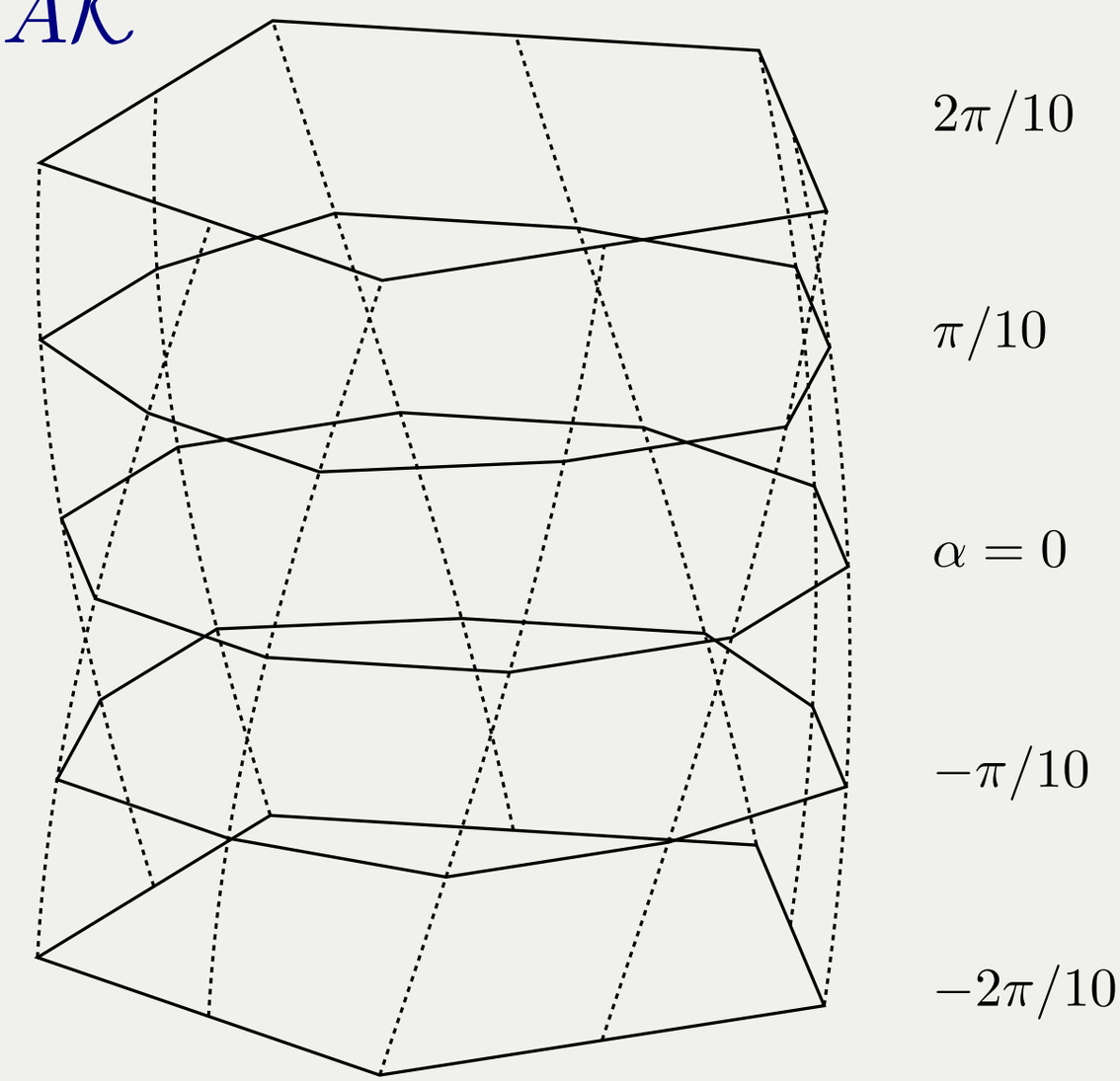
If

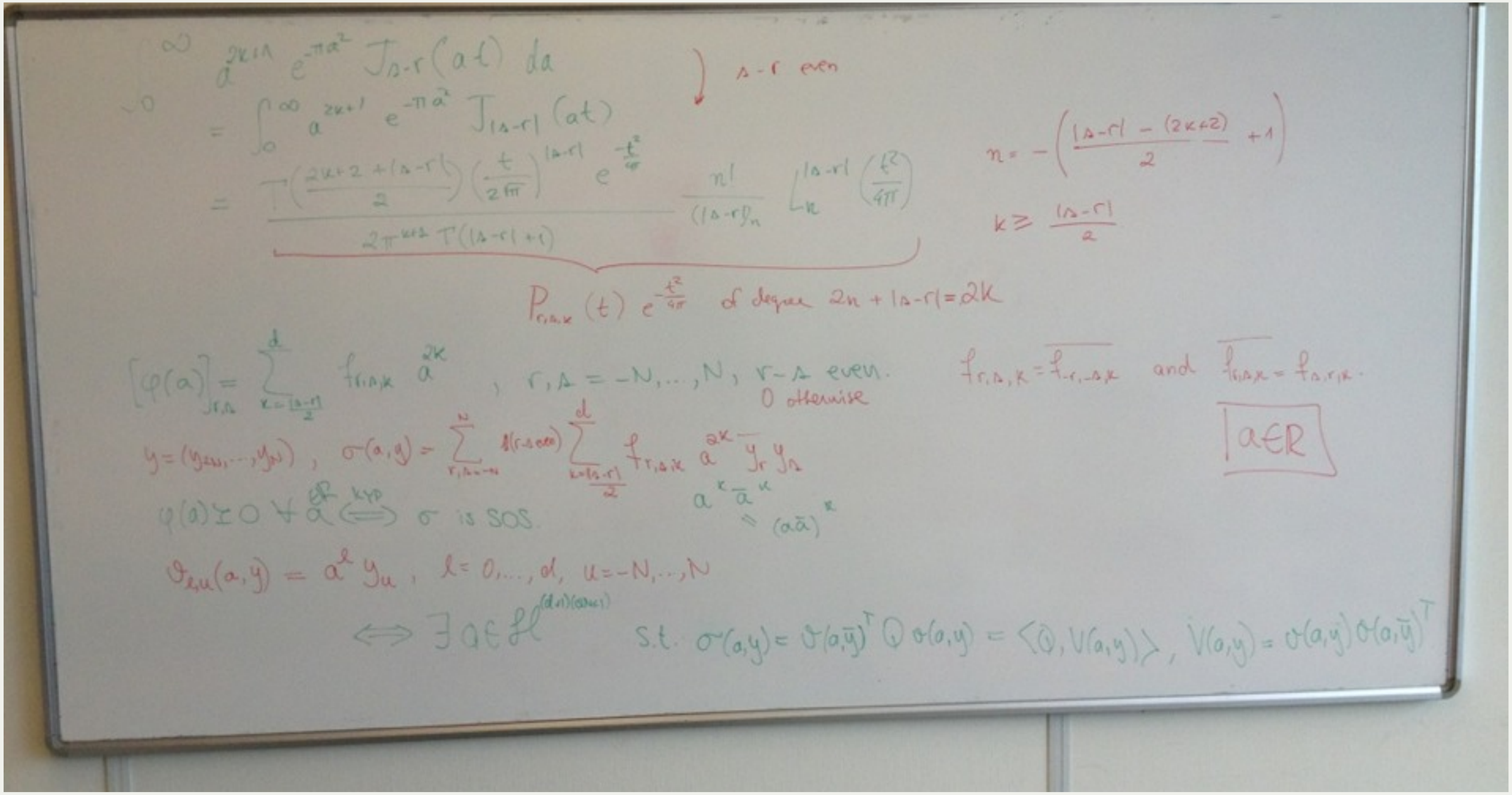
$$e^{\pi a^2} \sum_{r,s=-N}^{N} \hat{f}(a) y_r y_s \in \mathbb{R}[a, y_{-N}, \ldots, y_N]$$

is a sum of squares, then f is pos. type

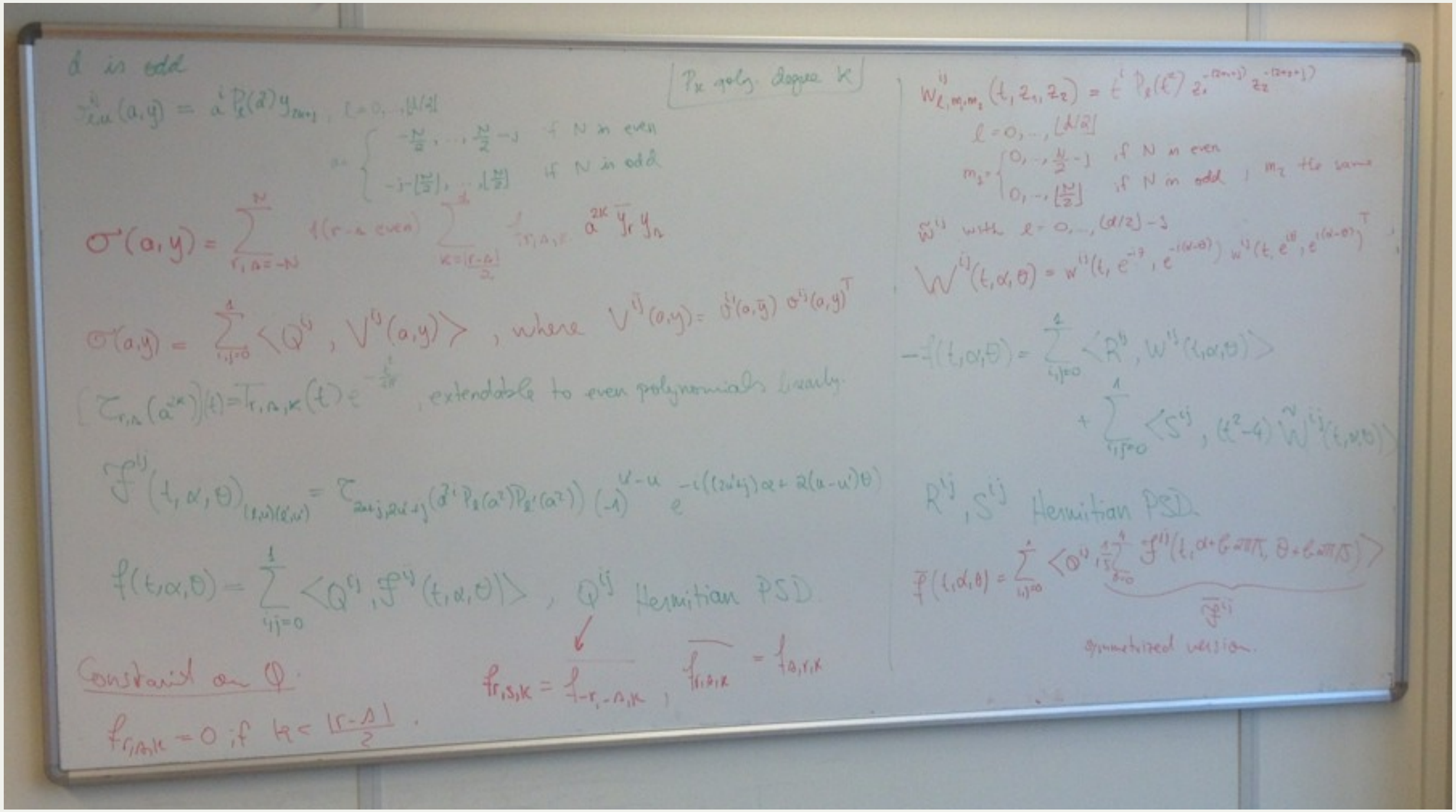
geometric condition

$$f(x, A) \leq 0 \text{ if } x \notin \mathcal{K} - A\mathcal{K}$$

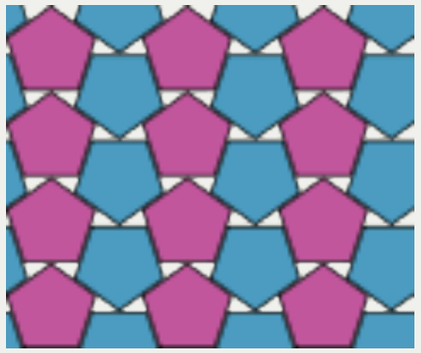




complete SDP (with only a few minor mistakes)



complete SDP (with only a few minor mistakes) continued

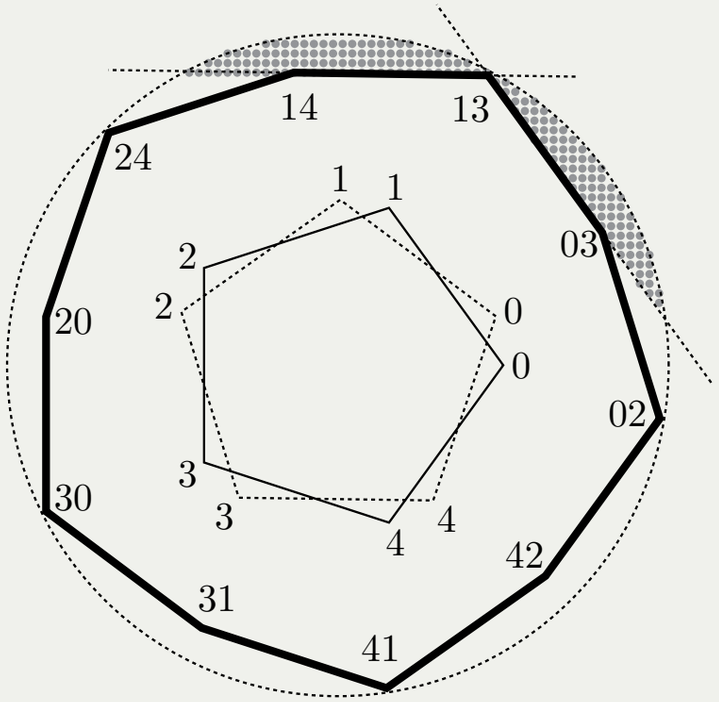


Kuperberg[2] (1992)

$\overline{\alpha} \in [0.92, ?]$

0.98 Oliveira, V. (2013)

- custom made C++ library for generating and analyzing SDPs with SOS constraints
- geometric constraint modeled by a mixture of sampling and SOS
- 0.98 can probably be improved



Improving Cohn-Elkies bound

de Laat, Oliveira, V. (2012)

1. Adding valid inequalities
   (bounds on average contact numbers)
2. More flexible numerical method

n	lower bound	Rogers	Cohn-Elkies	new bound
4	0.125	0.13127	0.13126	0.13081
5	0.08839	0.09987	0.09975	0.09955
6	0.07217	0.08112	0.08084	0.08070
7	0.0625	0.06981	0.06933	0.06926

density given as point density (= # centers per unit volume)

Rigorous computations

right choice of polynomial basis is extremely important

— using monomial basis fails badly, even for very small degrees

— our choice: $|\mu_k^{-1}| L_k^{n/2-1}(2\pi t)$

μ_k: coefficient of $L_k^{n/2-1}(2\pi t)$ with largest absolute value

— `csdp`: $d \leq 31$

— `SDPA-gmp` with 256 bits of precision: $d \leq 51$

In order to get mathematical rigorous results:

— perform post processing of the floating point solution

— perturb to a rational solution

— analyze quality-loss of this perturbation

(by estimates of eigenvalues and condition numbers)

Tetrahedra?

- ⋆ needs more automatization
  (also the harmonic analysis part)
- ⋆ needs more theory for numerical
  optimization with SOS constraints
  (condition numbers, special numerical solvers)
- ⋆ still a challenge

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